

FEM Mid-Term — Survival Sheet

Spring · Bar · Truss · Beam | 20 marks, Q1–Q4 | Goal: 10/20 to pass

Battle plan (read this first):

Each question (Spring/Bar/Truss/Beam) has sub-parts mixing theory + calculation. You don't need every part — just get the **easy theory sub-parts** right in all 4 questions and **fully nail the Spring problem** (it's the only "full solution" numeric — highest marks-per-minute). That alone can clear 10+.

Priority order if short on time: 1) Spring full solution (guaranteed marks) → 2) Theory one-liners (7 Qs, fast recall) → 3) Stiffness matrix derivations (write the setup even if you don't finish) → 4) Truss/Beam formulation steps (partial credit for correct setup, even without final numbers).

PART A — Theory (fast recall, 1 line each)

1. Experimental vs Numerical vs Analytical methods common Q

Method	What it is	Pros	Cons
Analytical	Exact closed-form solution of governing ODE/PDE	Exact, fast to evaluate	Only works for simple geometry/loading
Experimental	Build & physically test a prototype	Real-world truth, validates theory	Costly, slow, hard to vary parameters
Numerical (FEM)	Discretize domain, solve approximately	Handles any geometry/loading/material	Approximate, needs computing power, mesh-dependent

Trick: Remember as "**A-E-N = Ants Eat Nothing exactly**" — Analytical is exact, Experimental costs money, Numerical is the practical middle ground.

2. General FEM procedure common Q

1. Discretize the domain into elements (mesh)
2. Select displacement/shape (interpolation) functions for each element
3. Derive element stiffness matrix & load vector (energy method / Galerkin)
4. Assemble global stiffness matrix + global load vector
5. Apply boundary conditions
6. Solve the system $[K]\{u\} = \{F\}$ for nodal displacements
7. Post-process — get strains, stresses, reactions from displacements

Mnemonic: "**Dumb Students Don't Always Study, But Pass**" → Discretize, Shape functions, Derive, Assemble, apply Boundary conditions, Solve, Post-process.

3. FEM in software (ANSYS/COMSOL)

Pre-processing: build geometry → mesh it → assign material properties → apply loads & boundary conditions

Solution: software assembles global $[K]$, applies BCs, solves $\{u\} = [K]^{-1}\{F\}$
Post-processing: visualize – deformed shape, stress/strain contours, animations

4. Advantages & limitations of FEM

Advantages	Limitations
Handles complex geometry & loading	Approximate, not exact
Works for any material behaviour	Accuracy depends on mesh quality (convergence)
Cheaper than building prototypes	Computationally expensive for large models
Visual, easy to interpret results	"Garbage in, garbage out" — bad input $\Rightarrow$ bad output

5. Principle of Minimum Potential Energy (PMPE) also used in derivation!

Total Potential Energy: $\Pi = U - W$
 U = Strain energy stored = $(1/2) \int \sigma \epsilon \, dV$
 W = Work done by external forces

Statement: Of all admissible (kinematically consistent) displacement fields, the **true** displacement field is the one that makes total potential energy **stationary (minimum)** for stable equilibrium — i.e. $\partial \Pi / \partial u_i = 0$ for each DOF.

Trick: This is literally the engine behind Derivation Q1 (spring stiffness matrix) — so learn the statement *through* that derivation, don't memorize separately.

6. Discretization & boundary conditions

Discretization: Dividing a continuous body into a finite number of small elements connected at nodes, converting an infinite-DOF problem into a finite-DOF one.

Boundary conditions: Known values of displacement (essential/Dirichlet) or force (natural/Neumann) applied at the domain boundary — needed to get a unique, solvable $[K]\{u\}=\{F\}$ (otherwise $[K]$ is singular, since it doesn't include rigid body motion).

7. Global vs Local vs Natural coordinates

Type	What	Used for
Global	Fixed X-Y(-Z) system for the <i>whole structure</i>	Assembling elements, defining nodal coordinates
Local	Axis aligned with <i>one element's</i> own axis	Deriving that element's stiffness matrix simply
Natural	Dimensionless coordinate ξ (usually -1 to 1)	Shape functions & numerical (Gauss) integration, independent of element size

Trick: Global = "for the building", Local = "for one beam in it", Natural = "a universal ruler from -1 to 1 that fits any element."

PART B — Derivations (write the steps even if algebra gets messy — partial credit is real)

D1. Spring element stiffness matrix (via PMPE)

HIGH YIELD

Spring of stiffness k , nodes 1 & 2, displacements u_1, u_2 , nodal forces F_1, F_2 .

Strain energy: $U = (1/2) k (u_2 - u_1)^2$

Work done: $W = F_1 u_1 + F_2 u_2$

$\Pi = U - W = (1/2)k(u_2 - u_1)^2 - F_1 u_1 - F_2 u_2$

Minimize: $\partial \Pi / \partial u_1 = 0$ and $\partial \Pi / \partial u_2 = 0$

$\partial \Pi / \partial u_1 = -k(u_2 - u_1) - F_1 = 0 \Rightarrow k \cdot u_1 - k \cdot u_2 = F_1$

$\partial \Pi / \partial u_2 = k(u_2 - u_1) - F_2 = 0 \Rightarrow -k \cdot u_1 + k \cdot u_2 = F_2$

$\Rightarrow \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

Trick to remember shape: stiffness matrix is always *symmetric*, diagonal = $+k$, off-diagonal = $-k$, and **each row sums to zero** (rigid-body check — no strain if $u_1 = u_2$).

D2. Inclined truss element — global stiffness matrix

Let $l = \cos\theta$, $m = \sin\theta$ (direction cosines), L = length, A = area, E = Young's modulus.

Local (axial-only) stiffness: $k' = (AE/L) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (acts along bar axis only)

Global stiffness (4×4 , DOFs u_1, v_1, u_2, v_2):

$$K = (AE/L) \times \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Common mistake: forgetting the truss stiffness is only 4×4 because each node has 2 DOF (u, v) in global coords, even though the element itself only resists axial (1D) deformation.

D3. Transformation matrix for inclined truss element

Local axial displacement of a node = projection of its global (u, v) onto bar axis:

$$u' = l \cdot u + m \cdot v$$

So the transformation matrix $[T]$ (2×4) mapping global $\rightarrow$ local DOFs:

$$\{u_1'; u_2'\} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \{u_1; v_1; u_2; v_2\}$$

Then: $K_{\text{global}} = [T]^T [k'] [T]$ — this is exactly how D2's 4×4 matrix is generated from the 2×2 local one.

Trick: D2 and D3 are the SAME derivation asked two ways. Learn D3 (the T-matrix idea) properly and D2 falls out automatically as $K = T^T k T$.

PART C — Solved Problems

C1. Spring problem — FULL worked example

memorize this pattern

Problem: Node 1 (fixed wall) — spring $k_1 = 1000$ N/mm — Node 2 — spring $k_2 = 2000$ N/mm — Node 3 (free). A force $F = 3000$ N acts at Node 3. Find u_2, u_3 and the reaction at Node 1.

Step 1 — Element stiffness matrices:

$k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ for elements between nodes (1,2); $k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ for nodes (2,3)

Step 2 — Assemble global K (3×3):

$$K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \\ = \begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix}$$

Step 3 — Apply BC ($u_1 = 0$), reduce to 2×2 for u_2, u_3 :

$$\begin{bmatrix} 3000 & -2000 \\ -2000 & 2000 \end{bmatrix} \{u_2; u_3\} = \{0; 3000\}$$

Step 4 — Solve:

$$\text{Eq1: } 3000u_2 - 2000u_3 = 0 \Rightarrow u_2 = (2/3)u_3$$

$$\text{Eq2: } -2000u_2 + 2000u_3 = 3000$$

$$\Rightarrow -2000(2/3)u_3 + 2000u_3 = 3000 \Rightarrow (2000/3)u_3 = 3000$$

$$\Rightarrow u_3 = 4.5 \text{ mm}, u_2 = 3 \text{ mm}$$

Step 5 — Reaction at node 1:

$$R_1 = -k_1(u_2 - u_1) = -1000 \times 3 = -3000 \text{ N (i.e. 3000 N, balances applied F } \checkmark)$$

Trick: Always sanity-check: $\Sigma \text{Reactions} + \Sigma \text{Applied forces} = 0$. Here $-3000 + 3000 = 0$ $\checkmark$ — this check alone can catch 90% of arithmetic errors under exam pressure.

C2. Bar/Truss problem — formulation approach (partial marks version)

Typical ask: "Given a truss member from (0,0) to (3,4) m with A, E given — set up the global stiffness matrix" (full numeric solve usually NOT required — formulation is enough).

$$L = \sqrt{3^2 + 4^2} = 5 \text{ m}, \quad l = \cos\theta = 3/5 = 0.6, \quad m = \sin\theta = 4/5 = 0.8$$

Plug l , m directly into the D2 matrix template:

$$K = (AE/5) \times \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

Common mistake: mixing up l^2 and m^2 positions, or forgetting the matrix must be symmetric — if yours isn't symmetric, you made an arithmetic slip.

Temperature effect on stepped bars (also in syllabus): a change in temperature ΔT creates a thermal "load" even with no external force.

Thermal strain: $\epsilon_{th} = \alpha \cdot \Delta T$ (α = coefficient of thermal expansion)

Equivalent thermal nodal load: $\{F_{th}\} = AE\alpha \cdot \Delta T \times \{-1; 1\}$

Just add this to the normal load vector before solving — rest of the procedure (assemble, apply BC, solve) is identical to C1.

C3. Beam problem — formulation approach

Typical ask: set up the beam element stiffness matrix / assemble for a beam problem — again, formulation or partial calculation, not a full solve.

Beam element: 2 nodes, 2 DOF each (deflection v , rotation θ). For length L , E , I :

$$K = (EI/L^3) \times \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Trick to recall this matrix: "12, 6L, then halve-the-L-power-and-flip-sign-diagonally" — top-left 12 always positive, the two off-diagonal corners (-12) mirror it, and the $4L^2/2L^2$ alternate on the rotation DOFs.

Final 10-Minute Checklist

- Can you write PMPE statement + derive spring stiffness matrix cold? (covers Theory Q5 + Derivation D1)
- Can you solve C1 (spring, 3 nodes) from scratch without looking? — do it once by hand right now.
- Can you state the 7-step FEM procedure using the mnemonic?
- Do you know $l=\cos\theta$, $m=\sin\theta$ and can plug into the 4×4 truss matrix template?
- Do you recognize the beam matrix pattern ($12, 6L, 4L^2, 2L^2$) even if you can't derive it live?

If all 5 above are yes — you are safely past 10/20. Don't chase perfection on truss/beam full numerics; bank the guaranteed marks first, then attempt extra parts with remaining time. Good luck — go write it down once by hand before you walk in.